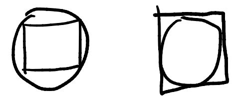
A subset A of $\mathbf{R}^n$ has (n-dimensional) **measure 0** if for every $\varepsilon > 0$ there is a cover $\{U_1, U_2, U_3, \ldots\}$ of A by closed or open rectangles such that $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$. It is obvious (but never-

countable

A subset A of $\mathbf{R}^n$ has (n-dimensional) **content 0** if for every $\varepsilon > 0$ there is a *finite* cover $\{U_1, \ldots, U_n\}$ of A by closed or open rectangles such that $\sum_{i=1}^{n} v(U_i) < \varepsilon$. If A has content 0,

Q: What if replace "rectangles" by "balls"? A: 

Thm A compact, then A has content 0 $\iff$ A has measure 0

Recall A is compat $\iff$ $\forall$ open cover $\{U_i\}_{i \in I}$, $\exists$ a finite subcover.

"$\Longrightarrow$": Clearly, even if A is not compact.

"$\Longleftarrow$": $\forall\, \varepsilon > 0$, $\exists$ an open rect cover $\{U_1, U_2 \cdots\}$ s.t. $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$

A is cpt $\Longrightarrow$ $\exists$ a subcover $\{U_{j_1}, U_{j_2} \cdots U_{j_n}\}$ of A

Clearly, $\sum_{k=1}^{n} v(U_{j_k}) \leq \sum_{i=1}^{\infty} v(U_i)$

Ex $\mathbb{R} \subset \mathbb{R}^2$ has measure 0 but doesn't have content 0

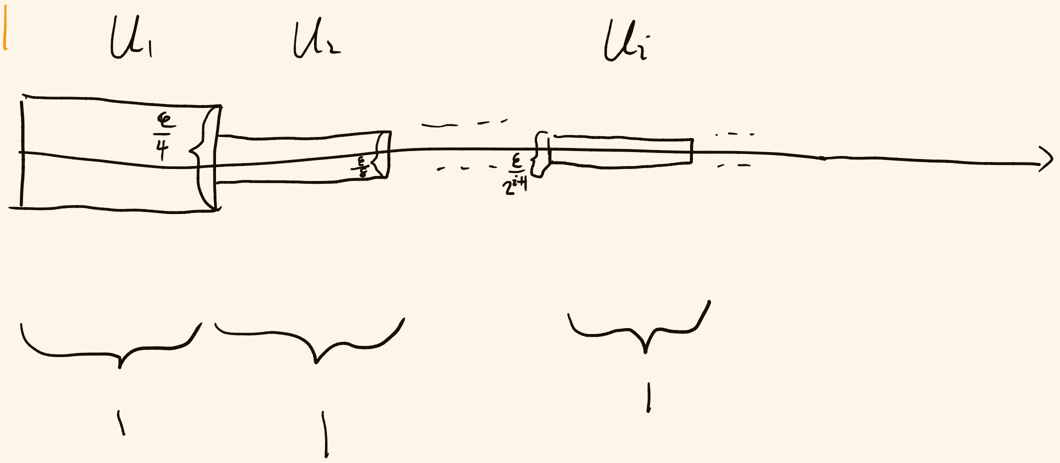
content: unbounded subsets don't have content 0.

measure: We show both $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$ have measure 0. Thus $\mathbb{R} = \mathbb{R}_{\geq 0} \cup \mathbb{R}_{\leq 0}$

have measure 0.

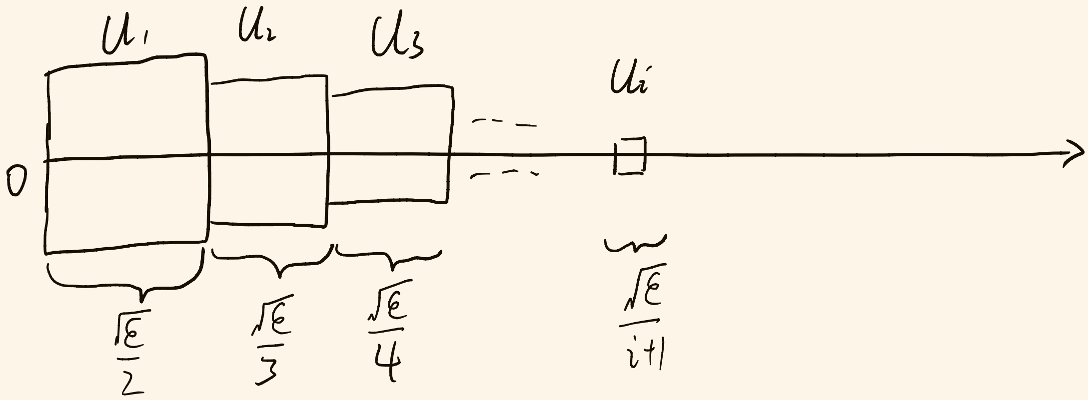
$\forall \varepsilon > 0$, Let $\{U_1, U_2 \cdots\}$ be the closed cover of $\mathbb{R}_{\geq 0}$ as shown:

Choice 1



$$\sum_{i=1}^{\infty} v(U_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2} < \varepsilon$$

or Choice 2



$$\sum_{i=1}^{\infty} v(U_i) = \sum_{i=1}^{\infty} \left(\frac{\sqrt{\varepsilon}}{i+1} \right)^2 = \sum_{i=1}^{\infty} \frac{\varepsilon}{(i+1)^2} = \left(\frac{\pi^2}{6} - 1\right)\varepsilon < \varepsilon$$

"Philosophy": Let $n < m$. "n-dimesional subsets" in $\mathbb{R}^m$ have measure 0.

Ex $S^n \subset \mathbb{R}^{n+1}$ non-Ex $\mathbb{Q} \subset \mathbb{R}$

1. Let f be a bounded integrable function on R. Prove that for any $\epsilon > 0$, there exists some $\delta > 0$ such that whenever $\mathcal{P}$ is a partition of R with $\operatorname{diam}(Q) < \delta$ for all $Q \in \mathcal{P}$, we have $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.

   Hint: Let $R = [a_1, b_1] \times \cdots [a_n, b_n]$ and $w = \max_i |b_i - a_i|$. For any partition $\mathcal{P}$ with diam$(Q) < \delta$ for all $Q \in \mathcal{P}$, if we take a refinement $\mathcal{P}'$ of $\mathcal{P}$ by adding one more grid point to $[a_i, b_i]$ for some i, then we have $L(f, \mathcal{P}') \leq L(f, \mathcal{P}) + 2M\delta w^{n-1}$ where $M > 0$ is a global bound for $|f|$.

   Solution. Let $\varepsilon > 0$. By Riemann condition, there exists a partition $\tilde{\mathcal{P}}$ of R such that $U(f, \tilde{\mathcal{P}}) - L(f, \tilde{\mathcal{P}}) < \varepsilon/2$. Let $\mathcal{P}$ be a partition of R with $\max\limits_{Q \in \mathcal{P}}(\operatorname{diam}(Q)) < \delta$. Following the hint, if $\mathcal{P}'$ is a common refinement of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ obtained by adding the grid points of $\tilde{\mathcal{P}}$ to $\mathcal{P}$, then

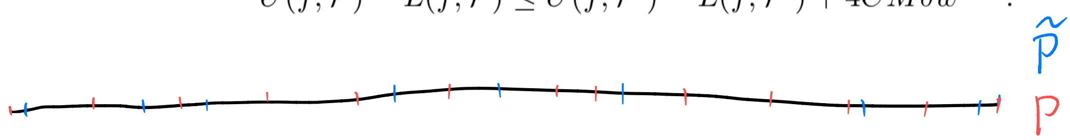
$$L(f, \mathcal{P}') \leq L(f, \mathcal{P}) + 2CM\delta w^{n-1},$$

   and similarly

$$U(f, \mathcal{P}') \geq U(f, \mathcal{P}) - 2CM\delta w^{n-1},$$

   where C is a constant that depends only on $\tilde{\mathcal{P}}$. Thus,

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) \leq U(f, \mathcal{P}') - L(f, \mathcal{P}') + 4CM\delta w^{n-1}.$$



$$U(f, \mathcal{P}) - L(f, \mathcal{P}) \leq \varepsilon/2 + 4CM\delta w^{n-1}.$$

The proof is complete by choosing a sufficiently small $\delta > 0$.

□

2. Let f be a bounded integrable function on R. Prove that for any $\epsilon > 0$, there exists some $\delta > 0$ such that whenever $\mathcal{P}$ is a partition of R with $\operatorname{diam}(Q) < \delta$ for all $Q \in \mathcal{P}$, and $x_Q \in Q$ is any arbitrarily chosen point inside $Q \in \mathcal{P}$, we have

$$\left| \sum_{Q \in \mathcal{P}} f(x_Q)\operatorname{Vol}(Q) - \int_R f \, dV \right| < \epsilon.$$

(The sum in the above expression is what we usually call the "Riemann sum"!)

Solution. It suffices to note that given a partition $\mathcal{P}$ and arbitrarily chosen points $x_Q \in Q$ for each $Q \in \mathcal{P}$, we have

$$L(f, \mathcal{P}) \leq \sum_{Q \in \mathcal{P}} f(x_Q)\operatorname{Vol}(Q) \leq U(f, \mathcal{P}),$$

and

$$L(f, \mathcal{P}) \leq \int_R f \, dV \leq U(f, \mathcal{P}).$$

□